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Letter to the Editor

On Best Simultaneous Approximation

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The purpose of this note is to prove a result on the existence and uniqueness of elements of best simultaneous approximation and to communicate that the two results proved by Sahney and Singh [6] are particular cases of the earlier proved results by Sastry and Naidu [7] and Sahney and Singh [5].

1. INTRODUCTION

Several mathematicians have studied the problem of best simultaneous approximation (cf. [1, 4]). Recently Sahney and Singh proved two results (cf. Theorems 1 and 2 of [5]) extending results of Holland *et al.* [3]. The uniqueness part of Theorem 1 of [5] is incorrect and the uniqueness part of Theorem 2 of [5] was given by the author in [1]. A more general form of Theorem 2 is available (cf. [7]). In this note we prove a result which corrects the existence part of Theorem 7 of [1] and the uniqueness part of Theorem 1 of [5]. We also note that the two results proved by Sahney and Singh [6] are particular cases of the earlier proved results by Sastry and Naidu [7] and Sahney and Singh [5].

2. BEST SIMULTANEOUS APPROXIMATION

DEFINITION 1. Let C be a subset of a normed linear space X. Given any bounded subset F of X, define

$$d(F, C) = \inf_{x \in C} \sup_{y \in F} \|y - x\|.$$

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An element x^* in C is said to be a *best simultaneous approximation* (b.s.a.) to F if

$$d(F, C) = \sup_{y \in F} \|y - x^*\|.$$

DEFINITION 2. A bounded subset F of a normed linear space X is said to be *remotal* with respect to a subset C of X if for each $x \in C$ there exists a point $f \in F$ farthest from x, i.e.,

$$\|x - f\| \geqslant \|x - y\| \qquad \text{for all } y \in F.$$

F is said to be *uniquely remotal* if such an f exists and is unique.

It is easy to see that every compact subset of a normed linear space is remotal with respect to the whole space.

The following two lemmas given in [3] will be used in the proof of our main theorem:

LEMMA 1. *Let C be any subset of a normed linear space X and F be a bounded subset of X. Then the mapping $\Phi: C \to \mathbb{R}$ defined by*

$$\Phi(x) = \sup_{y \in F} \|y - x\|$$

is continuous.

LEMMA 2. *Let C be a convex subset of a normed linear space X and F be any subset of X. If c_1 and c_2 are b.s.a. to F by elements of C then $\lambda c_1 + (1 - \lambda) c_2$, $0 \leqslant \lambda \leqslant 1$, is also a b.s.a. to F.*

The following theorem, which corrects the existence part of Theorem 7 of [1] and the uniqueness part of Theorem 1 of [5], gives the existence and uniqueness of elements of b.s.a.

THEOREM (Best Simultaneous Approximation Theorem). *Let X be a strictly convex normed linear space, C a compact convex subset of X and F be a subset of X which is remotal with respect to C. Then there exists a unique b.s.a. in C to F.*

Proof (Existence). Consider the function Φ defined previously. By Lemma 1, this function is continuous. Since C is compact, Φ attains its infimum at some $x^* \in C$, i.e.,

$$\sup_{y \in F} \|y - x^*\| = \Phi(x^*) = \inf_{x \in C} \Phi(x) = \inf_{x \in C} \sup_{y \in F} \|y - x\|.$$

This establishes the existence of an element of b.s.a.

Uniqueness: Suppose x_1^*, x_2^*, $x_1^* \neq x_2^*$ in C are two b.s.a. to the set F. i.e.,

$$\inf_{x \in C} \sup_{y \in F} \|x - y\| = \sup_{y \in F} \|y - x_1^*\| = \sup_{y \in F} \|y - x_2^*\| = r \qquad \text{(say)}. \tag{1}$$

By Lemma 2 and the convexity of C, $(x_1^* + x_2^*)/2 \in C$ is also an element of b.s.a. to F, i.e.,

$$\sup_{y \in F} \|y - (x_1^* + x_2^*)/2\| = r.$$

Since F is remotal with respect to C, there exists an element f^* in F such that

$$\|f^* - (x_1^* + x_2^*)/2\| = r. \tag{2}$$

Now (1) implies

$$\|f^* - x_1^*\| \leqslant r \qquad \text{and} \qquad \|f^* - x_2^*\| \leqslant r$$

and since the space is strictly convex, we have

$$\|f^* - (x_1^* + x_2^*)/2\| < r$$

unless $x_1^* = x_2^*$. This contradicts (2) and hence the uniqueness.

Remarks. (1) Uniqueness of the element of b.s.a. is also guaranteed if the function Φ defined above attains its infimum at exactly one $x^* \in C$.

(2) The unique remotality of F does not guarantee the uniqueness of element of b.s.a. as claimed by Sahney and Singh in [5, Theorem 1].

(3) Theorem 2 of [5] was proved in a more general form by Sastry and Naidu [7, Theorem 3].

(4) Using arguments similar to those of Theorem 2 of [5] or Theorem 1 of [7], it can be shown that the above theorem holds if C is a bounded weakly sequentially compact convex set.

Sahney and Singh proved the following two theorems in [6] on best simultaneous approximation:

THEOREM 1. *Let X be a strictly convex Banach space, and C a weakly compact, convex subset of X. Then there exists a unique best simultaneous approximation from the elements of C to any given compact subset F of X.*

THEOREM 2. *Let X be a strictly convex normed linear space and C a reflexive subspace of X. Then for any nonempty compact subset F of X there exists one and only one best simultaneous approximation in C.*

Since every compact set is remotal, Theorem 1 is a particular case of the following result proved by Sastry and Naidu in Theorem 3 of [7] (see also Remark (4) above).

THEOREM 3. *If X is a strictly convex normed linear space, K is boundedly weakly sequentially compact and convex and F is a farthest point set with respect to K, then there exists a unique best simultaneous approximation to F from K.*

Theorem 2 (also proved by Bosznay [2]) is a particular case of the following result proved by Sahney and Singh in Theorem 2 of [5].

THEOREM 4. *Let X be a strictly convex normed linear space and C a reflexive subspace of X. Then there exists one and only one best simultaneous approximation from the elements of C to any set F that is remotal with respect to C.*

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REFERENCES

1. G. C. AHUJA AND T. D. NARANG, On best simultaneous approximation, *Nieuw Arch. Wisk.* **27** (1979), 255–261.
2. A. P. BOSZNAY, A remark on simultaneous approximation, *J. Approx. Theory* **23** (1978), 296–298.
3. A. S. B. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Approx. Theory* **17** (1976), 187–188.
4. B. N. SAHNEY AND S. P. SINGH, On best simultaneous approximation, best simultaneous Chebyshev approximation with additive weight functions and related results, I, unpublished.
5. B. N. SAHNEY AND S. P. SINGH, On best simultaneous approximation, *in* "Approximation Theory III" (E. W. Cheney, Ed.), pp. 783–789, Academic Press, New York, 1980.
6. B. N. SAHNEY AND S. P. SINGH, On best simultaneous approximation in Banach spaces, *J. Approx. Theory* **35** (1982), 222–224.
7. K. P. R. SASTRY AND S. VENKATA RATNAM NAIDU, On best simultaneous approximation in normed linear spaces, *Proc. Nat. Acad. Sci. India Sect. A* **48** (4) (1978), 249–250.

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